

NON-ABELIAN HOPF COHOMOLOGY II

– THE GENERAL CASE –

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Abstract. We introduce and study non-abelian cohomology sets of Hopf algebras with coefficients in Hopf comodule algebras. We prove that these sets generalize as well Serre’s non-abelian group cohomology theory as the cohomological theory constructed by the authors in a previous article. We establish their functoriality and compute explicit examples. Further we classify Hopf torsors.

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INTRODUCTION. The present article, conceived as the continuation of [8], is devoted to the study of non-abelian cohomology theory in the Hopf algebra setting. We define a general cohomology theory analogous to that for groups adapted to Hopf algebras and suitable coefficient objects ([9], [10]). In order to clarify the purpose of our work, we recall first some basic facts about the classical constructions in the framework of groups. Let G be a group acting on a group A . The non-abelian cohomology theory $H^*(G, A)$ for groups may be organized in three different stages depending on the properties of the coefficient group A .

- Stage 1) *The group of coefficients A is abelian.* The classical Eilenberg-MacLane cohomology theory $H^*(G, A) = \text{Ext}_{\mathbf{Z}[G]}^*(\mathbf{Z}, A)$ produces a sequence of commutative groups. It provides useful invariants in homological algebra, algebraic topology and algebraic number theory.
- Stage 2) *The group of coefficients A is not abelian.* The previous construction fails in this case. However it is still possible to define a group $H^0(G, A)$ and a pointed set $H^1(G, A)$. This theory, called the non-abelian cohomology theory of groups, was introduced by Lang and Tate ([4]) for Galois groups with coefficients in an algebraic group, and was studied in full generality by Serre ([9], [10]). It is for instance well-known that the non-abelian cohomology set $H^1(G, A)$ classifies the G -torsors (or principal homogeneous spaces) on A (see [10]).
- Stage 3) *The group of coefficients A is the group of automorphisms of a G -Galois extension.* Suppose that the group G is finite and acts as a Galois group on a Galois extension S/R of noncommutative rings (for this generalization of Galois extensions of fields, see [5]). Let M be an S -module endowed with a compatible G -action. The latter induces a G -group structure on the group $\text{Aut}_S(M)$ of S -linear automorphisms of M . One of the authors ([7]) showed that in this context non-abelian cohomology theory comes into play. In particular he proved that the set $H^1(G, \text{Aut}_S(M))$ classifies objects which arise in descent theory along S/R , for example descent cocycles on M or twisted forms of M .

Hopf algebras naturally generalize groups. Kreimer and Takeuchi ([3]) widened Galois extensions to Hopf-Galois extensions of rings in the following spirit. As a group acting on rings plays the rôle of the symmetry object for Galois extensions, a Hopf algebra coacting on the rings does for Hopf-Galois extensions. In [8], we answered the natural question of extending Stage 3 to this setting. For a Hopf algebra H , an H -Hopf comodule algebra S , and an (H, S) -Hopf module M , we introduced a group $H^0(H, M)$ and a pointed set $H^1(H, M)$. This construction, here called *restricted non-abelian Hopf cohomology theory*, replaces $H^*(G, \text{Aut}_S(M))$. It offers a generalization of Stage 3 in the following two senses (see [8]):

- If S/R is an H -Hopf-Galois extension, then $H^1(H, M)$ classifies the analogue of descent cocycles on M along S/R and the twisted forms of M .
- Given a group G , a G -Galois extension of rings S/R is nothing but a \mathbf{Z}^G -Hopf-Galois extension, where \mathbf{Z}^G stands for the Hopf algebra of functions on G . Then $H^*(\mathbf{Z}^G, M)$ is isomorphic to $H^*(G, \text{Aut}_S(M))$.

The aim of this article is to define a non-abelian cohomology theory in the Hopf context corresponding to Stage 2. More precisely, let H be a Hopf algebra over a commutative ring k . For any H -comodule k -algebra E we introduce the *general non-abelian Hopf cohomology theory of H with coefficients in E* . We define a group $\mathcal{H}^0(H, E)$ and a pointed set $\mathcal{H}^1(H, E)$. These constructions are based on the non-abelian cohomology theory associated to a pre-cosimplicial group. We prove three main results (the precise wording and definitions will be found in the core of the article):

- (a) We show (Theorem 1.5) that the cohomology theory $\mathcal{H}^*(k^G, E)$ is isomorphic to $H^*(G, E^\times)$, where k^G denotes the Hopf algebra of functions on G and E^\times is the group of invertible elements of E .
- (b) Let S be an H -Hopf comodule algebra and M be an (H, S) -Hopf module. We establish (Theorem 2.6) that under lax technical conditions, $\mathcal{H}^*(H, \text{End}_S(M))$ and $H^*(H, M)$ are isomorphic.
- (c) Finally, if E is an H -comodule algebra, we classify (H, E) -Hopf torsors via the pointed set $\mathcal{H}^1(H, E)$ (Theorem 3.4).

The article is built in the following way. The first section is devoted to the definition and the properties of general non-abelian Hopf cohomology theory. There we prove Result (a), give some examples, explicit computations (§ 1.2 and § 1.4), and show that the Hopf module structures may be deformed with the help of 1-cocycles (Proposition 1.7). In § 1.6 we study the functoriality of the general non-abelian Hopf cohomology sets and write down an exact sequence associated to a sub-comodule algebra. In the second section, we clarify the links between general and restricted non-abelian Hopf cohomology theory. To this end, we state a technical condition (Condition (\mathcal{F}_n) in § 2.2) which allows to endow the endomorphism algebra of an Hopf module with a comodule structure (Lemma 2.4). We then deduce Result (b). The third and last section deals with Hopf torsors. We define them as a generalization of usual torsors (Definition 3.2, Proposition 3.7, and Corollary 3.8) and prove Result (c).

We mention here that an attempt of generalizing the non-abelian group cohomology theory to the Hopf context was done by Blanco Ferro ([1]). This author adapted Sweedler's theory ([11]), which can be viewed as a generalization of Stage 1. Blanco Ferro defined a 1-cohomology set $H^1(H, A)$, where H is a cocommutative Hopf algebra and A is an algebra not necessarily commutative. His construction is in some sense dual to ours. But if one tries to apply it to the Hopf-Galois extensions, one has to restrict oneself to a very particular case: not only does H have to be a commutative finitely generated k -projective Hopf algebra, but the Hopf-Galois extension S/k is over the ground field and moreover has to be commutative.

0. Conventions, notations, and terminology.

Let k be a fixed commutative and unital ring. The unadorned symbol \otimes between a right k -module and a left k -module stands for \otimes_k . By *(co-)algebra* we mean a (co-)unital (co-)associative k -(co-)algebra. By *(co-)module* over a (co-)algebra D , we always understand a right D -(co-)module unless otherwise stated. Let M be a k -module. We identify in a systematic way $M \otimes k$ with M .

For any algebra D , we denote by D^\times the group of invertible elements in D . If M is a D -module, $\text{End}_D(M)$ (respectively $\text{Aut}_D(M)$) is the algebra (respectively the group) of D -linear endomorphisms (respectively automorphisms) of M .

Let H be a Hopf algebra with multiplication μ_H , unity map η_H , comultiplication Δ_H , counity map ε_H , and antipode σ_H . Recall that an H -comodule algebra E is a k -module which is both an algebra and an H -comodule such that the coaction map is a morphism of algebras. A morphism of H -comodule algebras is simultaneously a morphism of algebras and of H -comodules. Suppose that E is an H -comodule algebra. Let M be both an E -module and an H -comodule. If the coaction map $\Delta_M : M \longrightarrow M \otimes H$ verifies the equality

$$\Delta_M(ms) = \Delta_M(m)\Delta_S(s)$$

for any $m \in M$ and $s \in E$, we say that M is an (H, E) -Hopf module (also called a *relative Hopf module* in the literature) and that Δ_M is (H, E) -linear. A *morphism of (H, E) -Hopf modules* is an E -linear map $f : M \longrightarrow M'$ such that $(f \otimes \text{id}_M) \circ \Delta_M = \Delta_{M'} \circ f$. Observe that E itself is naturally an (H, E) -Hopf module.

To denote the coactions on elements, we use the Sweedler-Heyneman convention, that is, for $m \in M$, we write $\Delta_M(m) = m_0 \otimes m_1$, with summation implicitly understood. More generally, when we write down a tensor we usually omit the summation sign \sum .

Let G be a finite group with neutral element e . Denote by k^G the k -free Hopf algebra over the k -basis $\{\delta_g\}_{g \in G}$, with the following structure maps: the multiplication is given by $\delta_g \cdot \delta_{g'} = \partial_{g, g'} \delta_g$, where $\partial_{g, g'}$ stands for the Kronecker symbol of g and g' ; the comultiplication Δ_{k^G} is defined by $\Delta_{k^G}(\delta_g) = \sum_{ab=g} \delta_a \otimes \delta_b$; the unit in k^G is the element $1 = \sum_{g \in G} \delta_g$; the counit ε_{k^G} is defined by $\varepsilon_{k^G}(\delta_g) = \partial_{g, e} 1$; the antipode σ_{k^G} sends δ_g on $\delta_{g^{-1}}$. When k is a field, then k^G is the dual of the usual group algebra $k[G]$.

1. General non-abelian Hopf cohomology theory.

The first section is devoted to the definition, the properties and examples of general non-abelian Hopf cohomology theory. The constructions are provided in simplicial terms (a résumé about the simplicial language may be found in [6]).

1.1. Definitions. Let $\mathcal{A}^* = A^0 \xrightarrow[d^1]{d^0} A^1 \xrightarrow[d^2]{d^1} A^2$ be a pre-cosimplicial group. The *non-abelian 0-cohomology group* $\mathbb{H}^0(\mathcal{A}^*)$ is the equalizer of the pair (d^0, d^1) :

$$\mathbb{H}^0(\mathcal{A}^*) = \{x \in A^0 \mid d^1(x) = d^0(x)\}.$$

The *non-abelian 1-cohomology pointed set* $\mathbb{H}^1(\mathcal{A}^*)$ is the right quotient

$$\mathbb{H}^1(\mathcal{A}^*) = A^0 \backslash \mathbb{Z}^1(\mathcal{A}^*).$$

Here the set $\mathbb{Z}^1(\mathcal{A}^*)$ of 1-cocycles is the subset of A^1 defined by

$$\mathbb{Z}^1(\mathcal{A}^*) = \{X \in A^1 \mid d^2(X)d^0(X) = d^1(X)\}.$$

The group A^0 acts on the right on A^1 by

$$X \leftarrow x = (d^1 x^{-1})X(d^0 x),$$

where $X \in A^1$ and $x \in A^0$. Using the pre-cosimplicial relations, one easily checks that this action restricts to $\mathbb{Z}^1(\mathcal{A}^*)$. Two 1-cocycles X and X' are said to be *cohomologous* if they belong to the same orbit under this action. The quotient set $\mathbb{H}^1(\mathcal{A}^*) = A^0 \backslash \mathbb{Z}^1(\mathcal{A}^*)$ is pointed with distinguished point the class of the neutral element of A^1 .

Let H be a Hopf algebra, let E be an H -comodule algebra with multiplication μ_E and coaction Δ_E . We define two maps $d^i : E \longrightarrow E \otimes H$ ($i = 0, 1$) and three maps $d^i : E \otimes H \longrightarrow E \otimes H \otimes H$ ($i = 0, 1, 2$) by the formulae

$$\begin{aligned} d^0(x) &= \Delta_E(x), & d^1(x) &= x \otimes 1, \\ d^0(X) &= (\Delta_E \otimes \text{id}_H)(X), & d^1(X) &= (\text{id}_E \otimes \Delta_H)(X), & d^2(X) &= X \otimes 1, \end{aligned}$$

where $x \in E$ and $X \in E \otimes H$.

Lemma 1.1. *The diagram $\mathcal{C}_{\leq 2}(H, E)$ given by*

$$\begin{array}{ccccc} E & \xrightarrow{d^0} & & \xrightarrow{d^0} & \\ & \searrow d^1 & E \otimes H & \xrightarrow[d^2]{d^1} & E \otimes H \otimes H \\ & & & & \end{array}$$

is a pre-cosimplicial object in the category of algebras.

Proof: The maps d^i are easily seen to be morphisms of algebras. The pre-cosimplicial relations $d^i d^j = d^j d^{i-1}$ for $i > j$ follow from the Hopf axioms for H and E . \square

Lemma 1.1 allows us to deduce a pre-cosimplicial diagram $\mathcal{C}_{\leq 2}^\times(H, E)$ in the category of groups by setting:

$$\begin{array}{ccccc} E^\times & \xrightarrow{d^0} & & \xrightarrow{d^0} & \\ & \searrow d^1 & (E \otimes H)^\times & \xrightarrow[d^2]{d^1} & (E \otimes H \otimes H)^\times \\ & & & & \end{array}$$

(we still denote by d^i the restrictions of the maps $d^i : E \otimes H^{\otimes j} \longrightarrow E \otimes H^{\otimes(j+1)}$ to the corresponding multiplicative groups).

Remark : Both $\mathcal{C}_{\leq 2}(H, E)$ and $\mathcal{C}_{\leq 2}^\times(H, E)$ are in fact cosimplicial objects. The codegeneracy maps on $\mathcal{C}_{\leq 2}(H, E)$ are given by

$$\begin{aligned} s^0 &= \text{id}_E \otimes \varepsilon_H : E \otimes H \longrightarrow E, \\ s^0 &= \text{id}_E \otimes \varepsilon_H \otimes \text{id}_H : E \otimes H \otimes H \longrightarrow E \otimes H \quad \text{and} \quad s^1 = \text{id}_E \otimes \text{id}_H \otimes \varepsilon_H : E \otimes H \otimes H \longrightarrow E \otimes H. \end{aligned}$$

The codegeneracy maps on $\mathcal{C}_{\leq 2}^\times(H, E)$ are again obtained by restriction.

Definition 1.2 : The general non-abelian Hopf cohomology objects $\mathcal{H}^*(H, E)$ of a Hopf algebra H with coefficients in an H -comodule algebra E is the non-abelian cohomology theory associated to the pre-cosimplicial diagram $\mathcal{C}_{\leq 2}^\times(H, E)$.

In other words

$$\begin{aligned}\mathcal{H}^0(H, E) &= \mathbb{H}^0(\mathcal{C}_{\leq 2}^\times(H, E)) = \{x \in E^\times \mid d^1(x) = d^0(x)\} \quad \text{and} \\ \mathcal{H}^1(H, E) &= \mathbb{H}^1(\mathcal{C}_{\leq 2}^\times(H, E)) = E^\times \setminus \mathcal{Z}^1(H, E).\end{aligned}$$

Observe that $\mathcal{H}^0(H, E)$ is the group $(E^{coH})^\times$ of invertible coinvariant elements of E . The set $\mathcal{Z}^1(H, E)$ of Hopf 1-cocycles of H with coefficients in E is the subset of $(E \otimes H)^\times$ given by

$$\mathcal{Z}^1(H, E) = \{X \in (E \otimes H)^\times \mid d^2(X)d^0(X) = d^1(X)\}.$$

We refer to $d^2(X)d^0(X) = d^1(X)$ as the Hopf 1-cocycle relation.

Remarks 1.3 :

a) For any Hopf algebra H and any H -comodule algebra E , one proves the inclusion

$$\mathcal{Z}^1(H, E) \subseteq \text{Ker}(\text{id}_E \otimes \varepsilon_H : (E \otimes H)^\times \longrightarrow E^\times)$$

by applying the map $\text{id}_E \otimes \varepsilon_H \otimes \text{id}_H$ to the Hopf 1-cocycle relation.

b) If the algebras E and H are both commutative, the sets $\mathcal{Z}^1(H, E)$ and $\mathcal{H}^1(H, E)$ become groups with product induced by the multiplication of $E \otimes H$.

1.2. First examples.

1) *The Hopf algebra is trivial.* Any algebra E is naturally a k -comodule algebra with the coaction Δ_E equal to id_E . One then has:

$$\mathcal{H}^0(k, E) = E^\times \quad \text{and} \quad \mathcal{H}^1(k, E) = \{1\}.$$

Indeed, the first equality is obvious. One checks that $\mathcal{Z}^1(k, E)$ is the pointed set of invertible idempotent elements of E , that is nothing else than $\{1\}$.

2) *The coefficients are trivial.* Let H be a Hopf algebra. The ground ring k is an H -comodule algebra through the coaction Δ_k given by the unity map η_H . Denote by $\text{Gr}(H)$ the group of grouplike elements in H . One then has:

$$\mathcal{H}^0(H, k) = k^\times \quad \text{and} \quad \mathcal{H}^1(H, k) \cong \text{Gr}(H),$$

the latter relation being an isomorphism of groups. The calculation of $\mathcal{H}^0(H, k)$ is straightforward. We compute now $\mathcal{Z}^1(H, k)$. A 1-cocycle is in particular an element $h \in H$ verifying the 1-cocycle relation, here $h \otimes h = \Delta_H(h)$. So the element h is grouplike, hence incidentally also invertible in H . The action of k^\times on $\mathcal{Z}^1(H, k)$ is trivial; therefore $\mathcal{H}^1(H, k)$ is the whole group of grouplike elements of H .

3) *The coefficients are the Hopf algebra itself.* A Hopf algebra H is a comodule algebra over itself. One has:

$$\mathcal{H}^0(H, H) = k^\times \quad \text{and} \quad \mathcal{H}^1(H, H) = \{1\}.$$

The first equality follows from the very definition: $\mathcal{H}^0(H, H) = (H^{coH})^\times$. To prove the second equality, pick $X \in \mathcal{Z}^1(H, H)$ and apply the map $\varepsilon_H \otimes \text{id}_H \otimes \text{id}_H$ to the cocycle relation $d^2(X)d^0(X) = d^1(X)$. One gets $(x \otimes 1)X = \Delta_H(x)$, with $x = (\varepsilon_H \otimes \text{id}_H)(X)$. So $\mathcal{Z}^1(H, H)$ is contained in the set $\{(x^{-1} \otimes 1)\Delta_H(x) \mid x \in H^\times\}$, which is equal to $\{d^1(x^{-1})d^0(x) \mid x \in H^\times\}$. Conversely, if $X = (x^{-1} \otimes 1)\Delta_H(x)$ for $x \in H^\times$, then X fulfills the cocycle relation. So $\mathcal{Z}^1(H, H)$ equals $\{d^1(x^{-1})d^0(x) \mid x \in H^\times\}$, and therefore the 1-cohomology set is trivial.

1.3. Link with non-abelian group cohomology. We first recall the definitions given by Serre ([9], [10]) of the non-abelian cohomology theory $H^i(G, A)$ (with $i = 0, 1$) of a group G with coefficients in a (left) G -group A . The 0-cohomology object $H^0(G, A)$ is the group A^G of invariant elements of A under the action of G . The set $Z^1(G, A)$ of 1-cocycles is given by

$$Z^1(G, A) = \{\alpha : G \longrightarrow A \mid \alpha(gg') = \alpha(g)^g(\alpha(g')), \quad \forall g, g' \in G\}.$$

It is pointed with distinguished point the constant map $1 : G \longrightarrow A$. The group A acts on the right on $Z^1(G, A)$ by

$$(\alpha \leftarrow a)(g) = a^{-1}\alpha(g)^g a,$$

where $a \in A$, $\alpha \in Z^1(G, A)$, and $g \in G$. Two 1-cocycles α and α' are *cohomologous* if they belong to the same orbit under this action. The non-abelian 1-cohomology set $H^1(G, A)$ is the left quotient $A \backslash Z^1(G, A)$. It is pointed with distinguished point the class of the constant map $1 : G \longrightarrow A$.

The non-abelian cohomology theory of groups may be interpreted as the non-abelian cohomology theory associated to the pre-cosimplicial diagram of groups

$$\mathcal{G}_{\leq 2}(G, A) = \left(A = \text{Map}(G^0, A) \xrightleftharpoons[d^1]{d^0} \text{Map}(G, A) \xrightleftharpoons[d^2]{d^1} \text{Map}(G^2, A) \right).$$

Here $\text{Map}(G^i, A)$ stands for the set of the maps from G^i to A , which is endowed with the group structure induced by pointwise multiplication. The coboundaries are given by

$$\begin{aligned} d^0(x) : g &\longmapsto {}^g x, & d^1(x) : g &\longmapsto x, \\ d^0(\alpha) : (g, g') &\longmapsto {}^g \alpha(g'), & d^1(\alpha) : (g, g') &\longmapsto \alpha(gg'), & d^2(\alpha) : (g, g') &\longmapsto \alpha(g), \end{aligned}$$

where $x \in A$, $g, g' \in G$ and $\alpha \in \text{Map}(G, A)$. The reader may easily check that the pre-cosimplicial relations are satisfied and that one has the equality

$$\mathbb{H}^*(\mathcal{G}_{\leq 2}(G, A)) = H^*(G, A).$$

We now connect the general non-abelian Hopf cohomology theory with the non-abelian cohomology theory of groups. Let G be a finite group and E be a k^G -comodule algebra. For any $x \in E$, write

$$\Delta_E(x) = \sum_{g \in G} {}^g x \otimes \delta_g.$$

This formula defines an action of the group G on the algebra E , hence on the group E^\times . One has the following result :

Proposition 1.4. *Let G be a finite group and E be a k^G -comodule algebra. The pre-cosimplicial groups $\mathcal{G}_{\leq 2}(G, E^\times)$ and $\mathcal{C}_{\leq 2}(k^G, E)$ are isomorphic.*

Before we give the proof, we state the following immediate consequence:

Theorem 1.5. *Let G be a finite group and E be a k^G -comodule algebra. There is the equality of groups*

$$\mathcal{H}^0(k^G, E) = H^0(G, E^\times)$$

and an isomorphism of pointed sets

$$\mathcal{H}^1(k^G, E) \cong H^1(G, E^\times).$$

Proof of Proposition 1.4. First remark that any element in $(E \otimes k^G)^\times$ is of the form $\sum_{g \in G} x_g \otimes \delta_g$, where for all $g \in G$, the element x_g belongs to E^\times . In the same way any element in $(E \otimes k^G \otimes k^G)^\times$ is of the form $\sum_{g, g' \in G} x_{g, g'} \otimes \delta_g \otimes \delta_{g'}$, where for all $g, g' \in G$, the element $x_{g, g'}$ belongs to E^\times . We consider the map $\gamma_* : \mathcal{C}_{\leq 2}(k^G, E) \longrightarrow \mathcal{G}_{\leq 2}(G, E^\times)$, given by

$$\begin{aligned} \gamma_0 &= \text{id}_{E^\times} \\ \gamma_1\left(\sum_{g \in G} x_g \otimes \delta_g\right) &: u \longmapsto x_u \\ \gamma_2\left(\sum_{g, g' \in G} x_{g, g'} \otimes \delta_g \otimes \delta_{g'}\right) &: (u, v) \longmapsto x_{u, v}, \end{aligned}$$

for any $u, v \in G$. On each level, γ_* is an isomorphism of groups since $(E \otimes k^G)^\times$ (respectively $(E \otimes k^G \otimes k^G)^\times$) is isomorphic to $(E^\times)^{|G|}$ (respectively to $(E^\times)^{|G|^2}$).

It remains to check that γ_* is a morphism of pre-cosimplicial objects, in other words γ_* verifies $\gamma_j d^i = d^i \gamma_{j-1}$ for any $1 \leq j \leq 2$ and $0 \leq i \leq j$. This is done by direct computations. For example, set

$$\nu = \gamma_2 d^0 \left(\sum_{g \in G} x_g \otimes \delta_g \right) = \gamma_2 \left(\sum_{g, g' \in G} g' x_g \otimes \delta_{g'} \otimes \delta_g \right).$$

So $\nu(u, v) = {}^u x_v$, for any $u, v \in g$. Hence $\nu = d^0 \gamma_1 \left(\sum_{g \in G} x_g \otimes \delta_g \right)$. As an other example, set

$$\nu' = \gamma_2 d^1 \left(\sum_{g \in G} x_g \otimes \delta_g \right) = \gamma_2 \left(\sum_{h, h' \in G} x_{hh'} \otimes \delta_h \otimes \delta_{h'} \right).$$

So $\nu'(u, v) = x_{uv}$, for any $u, v \in g$. Hence $\nu' = d^1 \gamma_1 \left(\sum_{g \in G} x_g \otimes \delta_g \right)$. We leave to the reader the three remaining computations. \square

Two direct applications of Theorem 1.5.

1) Let G be a finite group. One may recover the isomorphism between the group $\text{Gr}(k^G)$ of grouplike elements of k^G and the Pontryagin dual $\hat{G} = \text{Hom}(G, k^\times)$ of G . Indeed, by Example 2 of § 1.2, the group $\text{Gr}(k^G)$ is isomorphic to $\mathcal{H}^1(k^G, k)$. In this situation, the identification $\mathcal{H}^1(k^G, k) \cong H^1(G, k^\times)$ given by Theorem 1.5 is in fact an isomorphism of groups, and one sees that $H^1(G, k^\times)$ is isomorphic to \hat{G} .

2) For any finite subgroup G of a group L , the group ring $k[L]$ is canonically equipped with a k^G -comodule algebra structure $\Delta_{k[L]}$ given by $\Delta_{k[L]}(h) = \sum_{g \in G} ghg^{-1} \otimes \delta_g$, for any $h \in L$, and extended by linearity. Theorem 1.5 claims the isomorphism $\mathcal{H}^*(k^G, k[L]) \cong H^*(G, k[L]^\times)$. However the computation of the group of units in $k[L]$ is in general a very difficult problem: the group $k[L]^\times$ is known only for some particular groups L .

1.4. An explicit example where the Hopf algebra is not an algebra of functions on a group. Let here k be a field and H_4 be the Sweedler four-dimensional Hopf algebra over k . Recall that H_4 is generated by two elements g and h submitted to the relations:

$$g^2 = 1, \quad h^2 = 0, \quad gh + hg = 0.$$

On the generators, the comultiplication, the antipode, and the counit of H_4 are given by

$$\begin{aligned} \Delta(g) &= g \otimes g, & \Delta(h) &= h \otimes g + 1 \otimes h, \\ \sigma(g) &= g, & \sigma(gh) &= gh, \\ \varepsilon(g) &= 1, & \varepsilon(h) &= 0. \end{aligned}$$

Denote by E_2 the algebra of dual numbers, viewed as the subalgebra of H_4 generated by h . Via Δ , the algebra E_2 is naturally endowed with a structure of H_4 -comodule algebra.

Proposition 1.6. *There is an equality of groups*

$$\mathcal{H}^0(H_4, E_2) = k^\times$$

and an isomorphism of pointed sets

$$\mathcal{H}^1(H_4, E_2) \cong \{1 \otimes 1, 1 \otimes g\}.$$

Proof. The proof consists in calculating explicitly the invariants on the 0-level (we leave this point to the reader) and in writing down the cocycle relations on generic elements on the 1-level. The computation of $\mathcal{Z}^1(H_4, E_2)$ is lightened by remarking that $E_2 \otimes H_4 = \{1 \otimes U + h \otimes V \mid U, V \in H_4\}$ and that $H_4 = F \oplus Fh$, where F is the sub-Hopf algebra of H_4 generated by g . The cocycle relation is then equivalent to the following system of two conditions on $\Delta(U)$ and $\Delta(V)$:

$$\begin{cases} \Delta(U) = U \otimes U + U h \otimes V & (1) \\ \Delta(V) = (Ug) \otimes V + V \otimes U + (Vh) \otimes V. & (2) \end{cases}$$

In Equation (1), if one replaces U by $x + yh$ and V by $z + th$, with $x, y, z, t \in F$, one gets a system of four equations in x, y, z, t . Solving them, one deduces U and V , which automatically satisfy Equation (2).

Finally one obtains

$$\mathcal{Z}^1(H_4, E_2) = \{X_u, Y_u \mid u \in k\},$$

where the elements X_u and Y_u of $\mathcal{Z}^1(H_4, E_2)$ are given by

$$\begin{aligned} X_u &= 1 \otimes 1 + u(1 \otimes h) - u(h \otimes 1) + u(h \otimes g) - u^2(h \otimes h) \\ Y_u &= 1 \otimes g + u(1 \otimes gh) - u(h \otimes g) + u(h \otimes 1) - u^2(h \otimes gh). \end{aligned}$$

The distinguished point of $\mathcal{Z}^1(H_4, E_2)$ is $X_0 = 1 \otimes 1$. One may observe that $\mathcal{Z}^1(H_4, E_2)$ contains a group, the set $\{X_u \mid u \in k\}$, which acts on the right on $\mathcal{Z}^1(H_4, E_2)$ by way of the multiplication in $(E_2 \otimes H_4)^\times$. Indeed, for any $u, v \in k$ one has the formulae:

$$X_u X_v = X_{u+v}, \quad Y_u X_v = Y_{u+v}.$$

It remains to describe the action of E_2^\times on $\mathcal{Z}^1(H_4, E_2)$. A generic element in E_2^\times is of the form $\alpha + \beta h$, with $\alpha \in k^\times$ and $\beta \in k$. A direct computation gives the identities

$$X_u \leftarrow (\alpha + \beta h) = X_{u+\beta/\alpha} \quad \text{and} \quad Y_u \leftarrow (\alpha + \beta h) = Y_{u+\beta/\alpha},$$

from which we deduce the isomorphism $\mathcal{H}^1(H_4, E_2) \cong \{X_0, Y_0\} = \{1 \otimes 1, 1 \otimes g\}$. \square

1.5. *Deforming the Hopf module structure with a cocycle.* Let H be a Hopf algebra and E be an H -comodule algebra. We show how the natural structure of (H, E) -Hopf module on E may be deformed with the help of a Hopf cocycle. To this end, for any element X of $E \otimes H$, denote by Δ_E^X the map from E to $E \otimes H$ given on $x \in E$ by

$$\Delta_E^X(x) = X \Delta_E(x).$$

One has then the following result:

Proposition 1.7. *Let H be a Hopf algebra, E be an H -comodule algebra, and X be an element of $(E \otimes H)^\times$. Then*

- 1) *the element X is a Hopf 1-cocycle if and only if (E, Δ_E^X) is an (H, E) -Hopf module;*
- 2) *two Hopf 1-cocycles X and X' are cohomologous if and only if the (H, E) -Hopf modules (E, Δ_E^X) and $(E, \Delta_E^{X'})$ are isomorphic.*

Proof. 1) Let us prove that Δ_E^X defines a coaction on E if and only if X belongs to $\mathcal{Z}^1(H, E)$. Suppose that X is a Hopf 1-cocycle. We have to show the two identities $(\Delta_E^X \otimes \text{id}_H) \circ \Delta_E^X = (\text{id}_E \otimes \Delta_H) \circ \Delta_E^X$ and $(\text{id}_E \otimes \varepsilon_H) \circ \Delta_E^X = \text{id}_E$. Pick an element x in E . On the one hand, since Δ_E is a morphism of algebras, one has the equalities

$$\begin{aligned} ((\Delta_E^X \otimes \text{id}_H) \circ \Delta_E^X)(x) &= (\Delta_E^X \otimes \text{id}_H)(X \Delta_E(x)) \\ &= (X \otimes 1)((\Delta_E \otimes \text{id}_H)(X \Delta_E(x))) \\ &= ((X \otimes 1)((\Delta_E \otimes \text{id}_H)(X)))((\Delta_E \otimes \text{id}_H) \circ \Delta_E)(x). \end{aligned}$$

On the other hand, the following equalities hold:

$$\begin{aligned} ((\text{id}_E \otimes \Delta_H) \circ \Delta_E^X)(x) &= (\text{id}_E \otimes \Delta_H)(X \Delta_E(x)) \\ &= ((\text{id}_E \otimes \Delta_H)(X))((\text{id}_E \otimes \Delta_H) \circ \Delta_E)(x). \end{aligned}$$

Since $((\Delta_E \otimes \text{id}_H) \circ \Delta_E)(x)$ is equal to $((\text{id}_E \otimes \Delta_H) \circ \Delta_E)(x)$, it remains to remark that the identity $(X \otimes 1)((\Delta_E \otimes \text{id}_H)(X)) = (\text{id}_E \otimes \Delta_H)(X)$ is exactly the cocycle relation $d^2(X)d^0(X) = d^1(X)$. In a similar way, using Remark 1.3(a) and the identity $(\text{id}_E \otimes \varepsilon_H) \circ \Delta_E = \text{id}_E$, one proves the equality $(\text{id}_E \otimes \varepsilon_H) \circ \Delta_E^X = \text{id}_E$.

The map Δ_E is a morphism of algebras, whence for any x and x' in E , one has the equality $\Delta_E(xx') = \Delta_E(x)\Delta_E(x')$. So one gets $X\Delta_E(xx') = X\Delta_E(x)\Delta_E(x')$, or $\Delta_E^X(xx') = \Delta_E^X(x)\Delta_E^X(x')$. This proves that (E, Δ_E^X) is an (H, E) -Hopf module, where the E -module structure of E is still given by the multiplication.

Conversely, assume that Δ_E^X endows E with a structure of (H, E) -Hopf module. Applying the identity $(\Delta_E^X \otimes \text{id}_H) \circ \Delta_E^X = (\text{id}_E \otimes \Delta_H) \circ \Delta_E^X$ to the element $x = 1$, one obtains the cocycle relation for X .

2) Suppose now given two cohomologous Hopf 1-cocycles X and X' . Let x be an element of E^\times such that $X' = (d^1 x^{-1})X(d^0 x)$. One easily checks that $\tau_x : E \rightarrow E$, the left multiplication by x , realizes an isomorphism of (H, E) -Hopf module from $(E, \Delta_E^{X'})$ to (E, Δ_E^X) .

Conversely, assume that for two Hopf 1-cocycles X and X' , there exists an isomorphism of (H, E) -Hopf modules $\varphi : (E, \Delta_E^X) \rightarrow (E, \Delta_E^{X'})$. By E -linearity, φ is entirely determined by $\varphi(1)$, more precisely $\varphi = \tau_{\varphi(1)}$. Since φ is surjective, the element $\varphi(1)$ is invertible in E . The comodule compatibility relation $\Delta_E^{X'} \circ \varphi = (\varphi \otimes \text{id}_H) \circ \Delta_E^X$ then implies $d^1(\varphi(1))X = X'd^0(\varphi(1))$. \square

1.6. The cohomology exact sequence associated to a sub-comodule algebra. Let H be a Hopf algebra. By the very definition, any morphism $\varphi : D \longrightarrow E$ of H -comodule algebras gives rise to a group homomorphism $\mathcal{H}^0(\varphi) : \mathcal{H}^0(H, D) \longrightarrow \mathcal{H}^0(H, E)$ and to a morphism of pointed sets $\mathcal{H}^1(\varphi) : \mathcal{H}^1(H, D) \longrightarrow \mathcal{H}^1(H, E)$. Our aim is to produce an exact sequence in cohomology associated to any inclusion $\varphi : D \hookrightarrow E$ of H -comodule algebras. To this purpose, we state the following lemma, which is a slight generalization to the cosimplicial case of Serre's exact sequence enounced in the framework of non-abelian cohomology theory of groups.

Lemma 1.8. *Let $\varphi : \mathcal{A}^* \longrightarrow \mathcal{B}^*$ be an injective morphism of two pre-cosimplicial groups*

$$\mathcal{A}^* = A^0 \begin{array}{c} \xrightarrow{d^0} \\ \xrightarrow{d^1} \end{array} A^1 \begin{array}{c} \xrightarrow{d^0} \\ \xrightarrow{d^1} \\ \xrightarrow{d^2} \end{array} A^2 \quad \text{and} \quad \mathcal{B}^* = B^0 \begin{array}{c} \xrightarrow{d^0} \\ \xrightarrow{d^1} \end{array} B^1 \begin{array}{c} \xrightarrow{d^0} \\ \xrightarrow{d^1} \\ \xrightarrow{d^2} \end{array} B^2.$$

Let $\mathcal{C}^ = \mathcal{B}^* / \varphi(\mathcal{A}^*)$ be the pre-cosimplicial left quotient object in the category of pointed sets and let $\pi : \mathcal{B}^* \longrightarrow \mathcal{C}^*$ be the quotient map. Then there is an exact sequence of pointed sets*

$$1 \longrightarrow \mathbb{H}^0(\mathcal{A}^*) \xrightarrow{\mathbb{H}^0(\varphi)} \mathbb{H}^0(\mathcal{B}^*) \xrightarrow{\mathbb{H}^0(\pi)} \mathbb{H}^0(\mathcal{C}^*) \xrightarrow{\partial} \mathbb{H}^1(\mathcal{A}^*) \xrightarrow{\mathbb{H}^1(\varphi)} \mathbb{H}^1(\mathcal{B}^*).$$

Moreover, if $\varphi(A^i)$ is for $i = 0, 1, 2$ a normal subgroup of B^i , then the above exact sequence extends to the right in the following way:

$$1 \longrightarrow \mathbb{H}^0(\mathcal{A}^*) \xrightarrow{\mathbb{H}^0(\varphi)} \mathbb{H}^0(\mathcal{B}^*) \xrightarrow{\mathbb{H}^0(\pi)} \mathbb{H}^0(\mathcal{C}^*) \xrightarrow{\partial} \mathbb{H}^1(\mathcal{A}^*) \xrightarrow{\mathbb{H}^1(\varphi)} \mathbb{H}^1(\mathcal{B}^*) \xrightarrow{\mathbb{H}^1(\pi)} \mathbb{H}^1(\mathcal{C}^*).$$

Proof: The connecting morphism ∂ is obtained by usual diagram-chasing. We leave the reader check the functoriality of \mathbb{H}^* as well as the exactness of the two sequences. \square

We mention here that the definition of the non-abelian 0-cohomology object $\mathbb{H}^0(\mathcal{A}^*)$ as an equalizer does in fact not require any algebraic structure on the set A^0 . This observation leads to the following definition. For any inclusion of H -comodule algebras $\varphi : D \hookrightarrow E$, we introduce the *relative non-abelian 0-cohomology set*

$$\mathcal{H}^0(H, D \hookrightarrow E) = \mathbb{H}^0(\mathcal{C}_{\leq 2}^\times(H, E) / \mathcal{C}_{\leq 2}^\times(H, D)),$$

where $\mathcal{C}_{\leq 2}^\times(H, E) / \mathcal{C}_{\leq 2}^\times(H, D)$ is the pre-cosimplicial diagram of pointed sets

$$E^\times / D^\times \begin{array}{c} \xrightarrow{d^0} \\ \xrightarrow{d^1} \end{array} (E \otimes H)^\times / (D \otimes H)^\times \begin{array}{c} \xrightarrow{d^0} \\ \xrightarrow{d^1} \\ \xrightarrow{d^2} \end{array} (E \otimes H \otimes H)^\times / (D \otimes H \otimes H)^\times.$$

In the particular case where D^\times , $(D \otimes H)^\times$, and $(D \otimes H \otimes H)^\times$, are normal subgroups respectively of E^\times , $(E \otimes H)^\times$, and $(E \otimes H \otimes H)^\times$, then $\mathcal{C}_{\leq 2}^\times(H, E) / \mathcal{C}_{\leq 2}^\times(H, D)$ is a pre-cosimplicial group, and the definition

$$\mathcal{H}^1(H, D \hookrightarrow E) = \mathbb{H}^1(\mathcal{C}_{\leq 2}^\times(H, E) / \mathcal{C}_{\leq 2}^\times(H, D))$$

makes sense. Next result is a corollary of Lemma 1.8.

Proposition 1.9. *Let H be a Hopf algebra and $\varphi : D \longrightarrow E$ be an injective morphism of H -comodule algebras. The sequence of pointed sets*

$$1 \longrightarrow \mathcal{H}^0(H, D) \xrightarrow{\mathcal{H}^0(\varphi)} \mathcal{H}^0(H, E) \xrightarrow{\mathcal{H}^0(\pi)} \mathcal{H}^0(H, D \hookrightarrow E) \xrightarrow{\partial} \mathcal{H}^1(H, D) \xrightarrow{\mathcal{H}^1(\varphi)} \mathcal{H}^1(H, E)$$

is exact. Moreover, if D^\times , $(D \otimes H)^\times$, and $(D \otimes H \otimes H)^\times$ are normal subgroups respectively of E^\times , $(E \otimes H)^\times$, and $(E \otimes H \otimes H)^\times$, then the above exact sequence can be extended to the right in the following way:

$$1 \longrightarrow \mathcal{H}^0(H, D) \xrightarrow{\mathcal{H}^0(\varphi)} \mathcal{H}^0(H, E) \xrightarrow{\mathcal{H}^0(\pi)} \mathcal{H}^0(H, D \hookrightarrow E) \xrightarrow{\partial} \mathcal{H}^1(H, D) \xrightarrow{\mathcal{H}^1(\varphi)} \mathcal{H}^1(H, E) \xrightarrow{\mathcal{H}^1(\pi)} \mathcal{H}^1(H, D \hookrightarrow E).$$

2. Links between general and restricted non-abelian Hopf cohomology theory.

In this section, H is a Hopf algebra, S is an H -comodule algebra, and M is an (H, S) -Hopf module. In [8], we introduced a cohomology theory $H^*(H, M)$ that we qualify from now on as *restricted*. We first briefly recall its definition and then compare it to our general cohomology theory under some lax technical conditions.

2.1. Reminder on restricted non-abelian Hopf cohomology theory. As in [8], we endow the set $W_k^n(M) = \text{Hom}_k(M, M \otimes H^{\otimes n})$ with a k -algebra structure thanks to the composition-type product

$$\circ : W_k^n(M) \times W_k^n(M) \longrightarrow W_k^n(M)$$

given by

$$\begin{cases} \varphi \circ \varphi' = \varphi \circ \varphi' & \text{if } n = 0 \\ \varphi \circ \varphi' = (\text{id}_M \otimes \mu_H^{\otimes n}) \circ (\text{id}_M \otimes \chi_n) \circ (\varphi \otimes \text{id}_H^{\otimes n}) \circ \varphi' & \text{if } n > 0 \end{cases}$$

for $\varphi, \varphi' \in W_k^n(M)$; here $\chi_n : H^{\otimes n} \otimes H^{\otimes n} \longrightarrow (H \otimes H)^{\otimes n}$ denotes the intertwining operator defined by

$$\chi_n((a_1 \otimes \dots \otimes a_n) \otimes (b_1 \otimes \dots \otimes b_n)) = (a_1 \otimes b_1) \otimes \dots \otimes (a_n \otimes b_n).$$

Denote by $W_S^n(M)$ the subalgebra $\text{Hom}_S(M, M \otimes H^{\otimes n})$ of $W_k^n(M)$, where the S -module structure on $M \otimes H^{\otimes n}$ is given by $(m \otimes \underline{h})s = ms \otimes \underline{h}$, for any $m \in M$, $\underline{h} \in H^{\otimes n}$, and $s \in S$.

Let R be either the ground ring k or the algebra S . The algebras $W_R^0(M)$, $W_R^1(M)$ and $W_R^2(M)$ may be organized in a pre-cosimplicial diagram of monoids [8, Lemma 1.1]:

$$\mathcal{W}_{\leq 2}(H, M)_R = \left(W_R^0(M) \xrightleftharpoons[b^1]{b^0} W_R^1(M) \xrightleftharpoons[b^2]{b^1} W_R^2(M) \right).$$

The two maps $b^i : W_R^0(M) \longrightarrow W_R^1(M)$ ($i = 0, 1$) and the three maps $b^i : W_R^1(M) \longrightarrow W_R^2(M)$ ($i = 0, 1, 2$) are given for $\varphi \in W_R^0(M)$ and $\Phi \in W_R^1(M)$ by the formulae

$$\begin{aligned} b^0 \varphi &= (\text{id}_M \otimes \mu_H) \circ (\Delta_M \otimes \text{id}_H) \circ (\varphi \otimes \sigma_H) \circ \Delta_M \\ b^1 \varphi &= (\text{id}_M \otimes \eta_H) \circ \varphi \\ b^0 \Phi &= (\text{id}_M \otimes \mu_H \otimes \text{id}_H) \circ (\Delta_M \otimes T) \circ (\Phi \otimes \sigma_H) \circ \Delta_M \\ b^1 \Phi &= (\text{id}_M \otimes \Delta_H) \circ \Phi \\ b^2 \Phi &= (\text{id}_M \otimes \text{id}_H \otimes \eta_H) \circ \Phi = \Phi \otimes \eta_H, \end{aligned}$$

where T denotes the flip of $H \otimes H$ (i.e. the automorphism of $H \otimes H$ which sends an indecomposable tensor $h \otimes h'$ to $h' \otimes h$).

Recall the definitions stated in [8]. The restricted 0-cohomology group $H^0(H, M)$ is the equalizer $\{\varphi \in \text{Aut}_S(M) \mid b^1\varphi = b^0\varphi\}$ of the pair (b^0, b^1) . The restricted 1-cohomology set $H^1(H, M)$ is the quotient set $\text{Aut}_S(M) \backslash Z^1(H, M)$ of the set $Z^1(H, M)$ of restricted Hopf 1-cocycles of H with coefficients in M under the right action of the group $\text{Aut}_S(M)$. Recall that $Z^1(H, M)$ is the subgroup

$$Z^1(H, M) = \left\{ \Phi \in W_k^1(M) \mid \begin{array}{ll} (\text{ZC}_1) & \Phi(ms) = \Phi(m)s, \text{ for all } m \in M \text{ and } s \in S \\ (\text{ZC}_2) & (\text{id}_M \otimes \varepsilon_H) \circ \Phi = \text{id}_M \\ (\text{ZC}_3) & b^2\Phi \circ b^0\Phi = b^1\Phi \end{array} \right\}$$

of $W_S^1(M)$ and an element $f \in \text{Aut}_S(M)$ acts on the right on an element $\Phi \in Z^1(H, M)$ by

$$(\Phi \leftarrow f) = b^1 f^{-1} \circ \Phi \circ b^0 f.$$

We now give a new alternative description of $Z^1(H, M)$ which we shall need in the sequel.

Proposition 2.1: *The set $Z^1(H, M)$ may be written as*

$$Z^1(H, M) = \{\Phi \in W_S^1(M)^\times \mid b^2\Phi \circ b^0\Phi = b^1\Phi\}.$$

Proof: Let Φ be an element of $Z^1(H, M)$. First observe that Condition (ZC₁) means exactly that Φ belongs to $W_S^1(M)$. It suffices to prove that Condition (ZC₂) is equivalent to the \circ -invertibility of Φ under Condition (ZC₃). Set $F = \Phi \circ \Delta_M$. In the proof of Theorem 3.1 in [8], we showed that Φ satisfies (ZC₂) if and only if F satisfies Condition (CC₂), that is $(\text{id}_M \otimes \varepsilon_H) \circ F = \text{id}_M$. Similarly Φ fulfils (ZC₃) if and only if F fulfils Condition (CC₃), that is $(F \otimes \text{id}_H) \circ F = (\text{id}_M \otimes \Delta_H) \circ F$.

1) Suppose that Φ is invertible in $W_S^1(M)$ with inverse Φ' . Since the comultiplication map Δ_M is invertible in $W_k^1(M)$ with inverse $\Delta'_M = (\text{id}_M \otimes \sigma_H) \circ \Delta_M$, the map F is invertible in $W_k^1(M)$ with inverse $F' = \Delta'_M \circ \Phi'$. Compose both terms of the equality (CC₃) on the left with the map $\text{id}_M \otimes \text{id}_H \circ \varepsilon_H$. One gets $F = F \circ ((\text{id}_M \otimes \varepsilon_H) \circ F)$, which is equivalent to the relation $F = F \circ ((\text{id}_M \otimes \varepsilon_H) \circ F) \otimes \eta_H$. One may simplify by F , and one gets $((\text{id}_M \otimes \varepsilon_H) \circ F) \otimes \eta_H = \text{id}_{W_k^1(M)} = \text{id}_M \otimes \eta_H$. Applying now $\text{id}_M \otimes \varepsilon_H$ on the right, one obtains (CC₂).

2) Conversely, assume that Condition (CC₂) holds. We shall show that the map F' defined by $F' = (\text{id}_M \otimes \sigma_H) \circ F$ is the inverse of F in $W_k^1(M)$. We apply therefore $\text{id}_M \otimes (\mu_H \circ (\text{id}_H \otimes \sigma_H))$, respectively $\text{id}_M \otimes (\mu_H \circ (\sigma_H \otimes \text{id}_H))$, on the left to the equality (CC₃). We get $((\text{id}_M \otimes \varepsilon_H) \circ F) \otimes \eta_H = F \circ F'$, respectively $((\text{id}_M \otimes \varepsilon_H) \circ F) = F' \circ F$. By Condition (CC₂), this exactly means that F' is the inverse of F . The map Φ is therefore invertible in $W_k^1(M)$ with inverse

$$\Phi' = \Delta_M \circ F' = \Delta_M \circ ((\text{id}_M \otimes \sigma_H) \circ (\Phi \circ \Delta_M)).$$

It remains to show that Φ' is S -linear. For $m \in M$, we denote the tensor $\Phi(m) \in M \otimes H$ by $m_{[0]} \otimes m_{[1]}$. We have

$$\Phi'(m) = ((m_0)_{[0]})_0 \otimes ((m_0)_{[0]})_1 \sigma_H((m_0)_{[1]} m_1).$$

For any $s \in S$, we obtain

$$\begin{aligned}
\Phi'(ms) &= ((m_0 s_0)_{[0]})_0 \otimes ((m_0 s_0)_{[0]})_1 \sigma_H((m_0 s_0)_{[1]} m_1 s_1) \\
&= ((m_0)_{[0]} s_0)_0 \otimes ((m_0)_{[0]} s_0)_1 \sigma_H(s_1) \sigma_H((m_0)_{[1]} m_1) \\
&= ((m_0)_{[0]})_0 s_0 \otimes ((m_0)_{[0]})_1 s_1 \sigma_H(s_2) \sigma_H((m_0)_{[1]} m_1) \\
&= ((m_0)_{[0]})_0 s_0 \varepsilon_H(s_1) \otimes ((m_0)_{[0]})_1 \sigma_H((m_0)_{[1]} m_1) \\
&= \Phi'(m)s.
\end{aligned}$$

This computation, which proves the S -linearity of Φ' , uses the Hopf algebra yoga. Moreover the first and the third equalities come from $\Delta_M(ms) = \Delta_M(m)\Delta_S(s)$, whereas the second one is a consequence of the S -linearity of Φ . \square

Denote by

$$\mathcal{W}_{\leq 2}^\times(H, M) = \left(W_S^0(M) \times \xrightarrow[b^1]{b^0} W_S^1(M) \times \xrightarrow[b^2]{b^1} W_S^2(M) \times \right),$$

the pre-cosimplicial diagram of groups obtained by taking the \odot -invertible elements of $W_S^*(M)$. Proposition 2.1 leads us to state the following result:

Theorem 2.2: *Let H be a Hopf algebra, S be an H -comodule algebra, and M be an (H, S) -Hopf module. One has the equality*

$$H^*(H, M) = \mathbb{H}^*(\mathcal{W}_{\leq 2}^\times(H, M)).$$

2.2. Technical conditions. In this paragraph, we first point out technical conditions we shall need in the sequel in order to compare the general and the restricted non-abelian Hopf cohomology theories. We show then that these conditions are fulfilled in two natural cases.

For any $n \geq 0$, consider the linear map

$$\omega_n : \text{End}_S(M) \otimes H^{\otimes n} \longrightarrow W_S^n(M) = \text{Hom}_S(M, M \otimes H^{\otimes n})$$

given on an undecomposable tensor $f \otimes \underline{h} \in \text{End}_S(M) \otimes H^{\otimes n}$ by

$$\omega_n(f \otimes \underline{h})(m) = f(m) \otimes \underline{h},$$

where $m \in M$. Notice that ω_0 is the identity map of $\text{End}_S(M)$ and that, for any $n \geq 0$, the map ω_n is a morphism of algebras. For $n \geq 0$, we consider the following condition.

Condition (\mathcal{F}_n) : the map ω_n is an isomorphism of algebras.

By the very definitions, Condition (\mathcal{F}_0) always holds. The first natural case where Condition (\mathcal{F}_n) is satisfied for all $n \geq 0$ appears when H is a finitely generated free k -module. We develop now a second case.

Let $M^* = \text{Hom}_k(M, k)$ be the linear dual of the k -module M . Consider the evaluation map $d_M : M^* \otimes M \longrightarrow k$ given by $d_M(\nu \otimes m) = \nu(m)$, for any $\nu \in M^*$ and $m \in M$.

Proposition 2.3: Condition (\mathcal{F}_n) is satisfied for all $n \geq 0$ if the two following statements both hold:

- 1) the Hopf algebra H is free as a k -module or S is equal to the ground ring k ;
- 2) there exists a map $b_M : k \longrightarrow M \otimes M^*$, called birth-map, such that

$$(\text{id}_M \otimes d_M) \circ (b_M \otimes \text{id}_M) = \text{id}_M \quad \text{and} \quad (d_M \otimes \text{id}_{M^*}) \circ (\text{id}_{M^*} \otimes b_M) = \text{id}_{M^*}.$$

By convention, we set $b_M(1) = \sum_i e_i \otimes e^i$. With this notation, the previous two equalities are equivalent to

$$\sum_i e_i e^i(m) = m \quad \text{and} \quad \sum_i \nu(e_i) e^i = \nu,$$

for any $m \in M$ and $\nu \in M^*$.

Example: When M is a finitely generated free k -module with basis $(e_j)_{j=1,\dots,n}$ such a birth-map b_M exists and is given by $b_M(1) = \sum_{j=1}^n e_j \otimes e_j^*$. Here $(e_j^*)_{j=1,\dots,n}$ is the dual basis of $(e_j)_{j=1,\dots,n}$.

The data of a module together with an evaluation map and a birth-map abstracts the notion of duality in tensor categories (see [2]).

Proof of Proposition 2.3: First of all, we endow M^* with the left S -module structure given by

$$(s\nu)(m) = \nu(ms)$$

with $\nu \in M^*$, $m \in M$, and $s \in S$. The module $M \otimes M^* \otimes H^{\otimes n}$ becomes an algebra through the multiplication given on two elements $m \otimes \nu \otimes \underline{h}$ and $m' \otimes \nu' \otimes \underline{h}'$ of $M \otimes M^* \otimes H^{\otimes n}$ by the formula $(m \otimes \nu \otimes \underline{h})(m' \otimes \nu' \otimes \underline{h}') = \nu(m') m \otimes \nu \otimes \underline{h} \underline{h}'$. We introduce the subalgebra $E_S^n(M)$ of $M \otimes M^* \otimes H^{\otimes n}$ consisting of the elements $m \otimes \nu \otimes \underline{h}$ such that, for any $s \in S$, one has $ms \otimes \nu \otimes \underline{h} = m \otimes s\nu \otimes \underline{h}$. Notice that under the first statement, one has the equality

$$E_S^n(M) = E_S^0(M) \otimes H^{\otimes n}.$$

We show now that, under the second statement, $E_S^n(M)$ is isomorphic to $W_S^n(M)$ as an algebra. First observe that the existence of a birth-map allows to write the action of s on ν as $s\nu = \sum_i \nu(e_i s) e^i$. Moreover one has $\sum_i e_i s \otimes e^i = \sum_i e_i \otimes s e^i$, in other words, $b_M(1)$ belongs to $E_S^0(M)$.

Consider the morphism $\lambda_n : E_S^n(M) \longrightarrow W_S^n(M)$ defined by

$$(\lambda_n(m \otimes \nu \otimes \underline{h}))(m') = \nu(m') m \otimes \underline{h},$$

with $m, m' \in M$, $\nu \in M^*$, and $\underline{h} \in H^{\otimes n}$. One checks that λ_n is well-defined with respect to the S -invariance and that it is a morphism of algebras. We prove now that under the existence of a birth-map, λ_n is a bijection. Let us explicit the inverse map. Denote by $\lambda'_n : W_S^n(M) \longrightarrow M \otimes M^* \otimes H^{\otimes n}$ the map given on an element $\Phi \in W_S^n(M)$ by

$$\lambda'_n(\Phi) = \sum_i \Phi(e_i)_0 \otimes e^i \otimes \Phi(e_i)_1,$$

where, for any $m \in M$, we set $\Phi(m) = \Phi(m)_0 \otimes \Phi(m)_1 \in M \otimes H^{\otimes n}$.

The map λ'_n takes its values in $E_S^n(M)$. Indeed, using the S -linearity of $\Phi \in W_S^n(M)$ and the fact that $b_M(1)$ belongs to $E_S^0(M)$, we have, for any $s \in S$:

$$\sum_i \Phi(e_i)_0 \otimes s e^i \otimes \Phi(e_i)_1 = \sum_i \Phi(e_i s)_0 \otimes e^i \otimes \Phi(e_i s)_1 = \sum_i \Phi(e_i)_0 s \otimes e^i \otimes \Phi(e_i)_1.$$

Moreover the map λ'_n is a morphism of algebras: for $\Phi, \Psi \in W_S^n(M)$, one has

$$\begin{aligned} \lambda'_n(\Phi) \lambda'_n(\Psi) &= \sum_{i,j} e^i(\Psi(e_j)_0) \Phi(e_i)_0 \otimes e^j \otimes \Phi(e_i)_1 \Psi(e_j)_1 \\ &= \sum_{i,j} \Phi(e^i(\Psi(e_j)_0) e_i)_0 \otimes e^j \otimes \Phi(e^i(\Psi(e_j)_0) e_i)_1 \Psi(e_j)_1 \\ &= \sum_j \Phi(\Psi(e_j)_0)_0 \otimes e^j \otimes \Phi(\Psi(e_j)_0)_1 \Psi(e_j)_1 \\ &= \sum_j (\Phi \circ \Psi)(e_j)_0 \otimes e^j \otimes (\Phi \circ \Psi)(e_j)_1 \\ &= \lambda'_n(\Phi \circ \Psi) \end{aligned}$$

It remains to compute the two compositions $\lambda_n \circ \lambda'_n$ and $\lambda'_n \circ \lambda_n$. One has, for any $\Phi \in W_S^n(M)$ and $m \in M$:

$$\lambda_n(\lambda'_n(\Phi))(m) = \sum_i e^i(m) \Phi(e_i)_0 \otimes \Phi(e_i)_1 = \Phi\left(\sum_i e^i(m) e_i\right) = \Phi(m).$$

On the other hand, for $m \otimes \nu \otimes \underline{h} \in M \otimes M^* \otimes H^{\otimes n}$, one obtains

$$\lambda'_n(\lambda_n(m \otimes \nu \otimes \underline{h})) = \sum_i \nu(e_i) m \otimes e^i \otimes \underline{h} = m \otimes \left(\sum_i \nu(e_i) e^i\right) \otimes \underline{h} = m \otimes \nu \otimes \underline{h}.$$

To end the proof, we write down the following sequence of isomorphisms, the composition of which is ω_n :

$$\text{End}_S(M) \otimes H^{\otimes n} = W_S^0(M) \otimes H^{\otimes n} \xrightarrow{\lambda_0 \otimes \text{id}_H^{\otimes n}} E_S^0(M) \otimes H^{\otimes n} = E_S^n(M) \xrightarrow{\lambda_n^{-1}} W_S^n(M).$$

Hence ω_n is an isomorphism of algebras, i.e. Condition (\mathcal{F}_n) is fulfilled. \square

2.3. An H -comodule structure on $\text{End}_S(M)$. Suppose from now on that Condition (\mathcal{F}_n) is satisfied for $0 \leq n \leq 2$. We define the morphism $\Delta_{\text{End}_S(M)} : \text{End}_S(M) \longrightarrow \text{End}_S(M) \otimes H$ to be the composition map

$$\text{End}_S(M) = W_S^0(M) \xrightarrow{b^0} W_S^1(M) \xrightarrow{\omega_1^{-1}} \text{End}_S(M) \otimes H.$$

Lemma 2.4: *The map $\Delta_{\text{End}_S(M)}$ endows $\text{End}_S(M)$ with a structure of H -comodule algebra.*

Proof: As a composition of morphisms of algebras, $\Delta_{\text{End}_S(M)}$ is a morphism of algebras. Let us prove that $\Delta_{\text{End}_S(M)}$ is coassociative. To this end, consider the following diagram in which the upper horizontal and the left vertical compositions are $\Delta_{\text{End}_S(M)}$:

$$\begin{array}{ccccccc}
 \text{End}_S(M) & \xlongequal{\quad} & W_S^0(M) & \xrightarrow{b^0} & W_S^1(M) & \xrightarrow{\omega_1^{-1}} & \text{End}_S(M) \otimes H \\
 \parallel & & \parallel & & \parallel & & \downarrow \omega_1 \\
 W_S^0(M) & & & & W_S^1(M) & & \\
 \downarrow b^0 & & & & \downarrow b^0 & & \\
 W_S^1(M) & & & & W_S^2(M) & & \\
 \downarrow \omega_1^{-1} & & & & \downarrow \omega_2^{-1} & & \\
 \text{End}_S(M) \otimes H & \xrightarrow{\omega_1} & W_S^1(M) & \xrightarrow{b^1} & W_S^2(M) & \xrightarrow{\omega_2^{-1}} & \text{End}_S(M) \otimes H^{\otimes 2}
 \end{array}$$

The pre-cosimplicial relation $b^0 b^0 = b^1 b^0$ implies the commutativity of the inner octogon, hence of the whole diagram. One may see that the lower horizontal composition is $\text{id}_{\text{End}_S(M)} \otimes \Delta_H$ and that the right vertical composition is $\Delta_{\text{End}_S(M)} \otimes \text{id}_H$. This shows the coassociativity of $\Delta_{\text{End}_S(M)}$.

The compatibility with the counit $(\text{id}_{\text{End}_S(M)} \otimes \varepsilon_H) \circ \Delta_{\text{End}_S(M)} = \text{id}_{\text{End}_S(M)}$ is a consequence of the relation $(\text{id}_{\text{End}_S(M)} \otimes \varepsilon_H) \circ b^0(\varphi) = \varphi$, which holds for all $\varphi \in \text{End}_S(M)$. \square

This construction allows us to define the cohomology of the Hopf algebra H with values in the H -comodule algebra $\text{End}_S(M)$. So, under the hypothesis that Condition (\mathcal{F}_n) is satisfied for $0 \leq n \leq 2$, the cohomology sets $\mathcal{H}^i(H, \text{End}_S(M))$ ($i = 0, 1$) make sense.

2.4. The Comparison Theorem. We are now able to compare restricted and general non-abelian Hopf cohomology theories.

Proposition 2.5: *Let H be a Hopf algebra, S be an H -comodule algebra, and M be an (H, S) -Hopf module such that Condition (\mathcal{F}_n) is satisfied for $0 \leq n \leq 2$. The pre-cosimplicial groups $\mathcal{C}_{\leq 2}^\times(H, M)$ and $\mathcal{W}_{\leq 2}^\times(H, \text{End}_S(M))$ are isomorphic.*

Proof: The map ω_n is an isomorphism of algebras since Condition (\mathcal{F}_n) holds. Moreover, one checks the equalities

$$\omega_j d^i = b^i \omega_{j-1}$$

for any $1 \leq j \leq 2$ and $0 \leq i \leq j$. \square

Theorem 2.2 and Proposition 2.5 imply the following result:

Theorem 2.6: *Let H be a Hopf algebra, S be an H -comodule algebra, and M be an (H, S) -Hopf module such that Condition (\mathcal{F}_n) is satisfied for $0 \leq n \leq 2$. Then there is an equality of groups*

$$\mathcal{H}^0(H, \text{End}_S(M)) = \mathcal{H}^0(H, M)$$

and an isomorphism of pointed sets

$$\mathcal{H}^1(H, \text{End}_S(M)) \cong \mathcal{H}^1(H, M).$$

3. Hopf torsors.

In this section, we define Hopf torsors. They generalize the classical torsors used in the framework of groups. We show that Hopf torsors are classified by a general non-abelian Hopf 1-cohomology set.

3.1. Definition of Hopf torsors. Let E be an algebra and T be a left E -module. For any $u \in T$, consider the E -linear map $\vartheta_u : E \longrightarrow T$ defined on $x \in E$ by $\vartheta_u(x) = ux$. Denote by T^\times the set

$$T^\times = \{u \in T \mid \vartheta_u \text{ is bijective}\}.$$

From now on, we deal with E -modules T such that T^\times is not empty. For example, if T is E itself the above set coincides with the group E^\times of invertible elements of the algebra E . Moreover, for any E -module T , observe that T^\times inherits the structure of an E^\times -set. In the following lemma, we collect several technical results about T^\times .

Lemma 3.1: *Let E be an algebra and T be a left E -module such that the set T^\times is not empty.*

- 1) *Let u be an element of T^\times . Then $\vartheta_u^{-1}(v)$ is, for any $v \in T$, the unique element of E such that $v\vartheta_u^{-1}(v) = v$.*
- 2) *Let v and v' be two elements in T and u be an element in T^\times . Then one has the identity $\vartheta_u^{-1}(v)\vartheta_u^{-1}(v') = \vartheta_u^{-1}(v\vartheta_u^{-1}(v'))$.*
- 3) *For any $u \in T^\times$, the map ϑ_u realizes a bijection between E^\times and T^\times .*

Proof: The first point is a direct consequence of the definition of ϑ_u . To show the second point, one writes $u\vartheta_u^{-1}(v)\vartheta_u^{-1}(v') = v\vartheta_u^{-1}(v') = u\vartheta_u^{-1}(v\vartheta_u^{-1}(v'))$, and concludes by uniqueness. Let us prove the third point. We have to show that, for any $u \in T^\times$, the bijection $\vartheta_u : E \longrightarrow T$ restricts to a bijection between E^\times and T^\times . For any $u \in T^\times$, the set $\vartheta_u(E^\times)$ is contained in T^\times . Indeed if x belongs to E^\times , one has $\vartheta_{ux} = \vartheta_u \circ \tau_x$, where τ_x denotes the left multiplication by x , which is bijective. The induced map remains injective. To prove that it is surjective, it is sufficient to show that $\vartheta_u^{-1}(v)$ belongs to E^\times for any $v \in T^\times$. By point 2), one has $v\vartheta_u^{-1}(v)\vartheta_v^{-1}(u) = v\vartheta_v^{-1}(u) = u$, so $\vartheta_u^{-1}(v)\vartheta_v^{-1}(u) = 1$. \square

Let H be a Hopf algebra, E be an H -comodule algebra, and (T, Δ_T) be an (H, E) -Hopf module. In this situation, the tensor product $T \otimes H$ is an $E \otimes H$ -module and $(T \otimes H)^\times$ makes sense. Notice that if u belongs to T^\times , then $u \otimes 1$ lies in $(T \otimes H)^\times$, since $\vartheta_{u \otimes 1} = \vartheta_u \otimes \text{id}_H$. In particular, if T^\times is non-empty, so is $(T \otimes H)^\times$.

We introduce now the set

$$T^\bullet = \{u \in T^\times \mid \Delta_T(u) \in (T \otimes H)^\times\}.$$

Definition 3.2: Let H be a Hopf algebra, E be an H -comodule algebra. An (H, E) -Hopf torsor is an (H, E) -Hopf module (T, Δ_T) such that the set T^\bullet is non-empty.

In particular E is an (H, E) -Hopf torsor. Indeed E is an (H, E) -Hopf module and Δ_E being a morphism of algebras, Δ_E sends any element of E^\times into $(E \otimes H)^\times$.

We denote by $\text{tors}(H, E)$ the set of (H, E) -Hopf torsors. It is pointed with distinguished point (E, Δ_E) . Two (H, E) -torsors (T, Δ_T) and $(T', \Delta_{T'})$ are *equivalent* if T and T' are isomorphic as (H, E) -Hopf modules. We denote by $\text{Tors}(H, E)$ the set of equivalence classes of (H, E) -torsors; it is pointed with distinguished point the class of (E, Δ_E) .

Lemma 3.3: *Let T be an (H, E) -Hopf torsor. Then the sets T^\bullet and T^\times coincide.*

Proof: Pick v in T^\times and u in T^\bullet . One has $v = u\vartheta_u^{-1}(v)$, thus $\Delta_T(v) = \Delta_T(u)\Delta_E(\vartheta_u^{-1}(v))$. By definition, the term $\Delta_T(u)$ belongs to $(T \otimes H)^\times$, and the factor $\Delta_E(\vartheta_u^{-1}(v))$ is invertible in $E \otimes H$ since Δ_E is a morphism of algebras. In the same way as E^\times acts on T^\times , the group $(E \otimes H)^\times$ acts on $(T \otimes H)^\times$, hence $\Delta_T(v)$ is an element of $(T \otimes H)^\times$, in other words v belongs to T^\bullet . \square

3.2. The non-abelian 1-Hopf cohomology set and Hopf torsors. As in the world of groups, the Hopf torsors are classified by a non-abelian 1-cohomology set. We detail this point now.

Theorem 3.4: *Let H be a Hopf algebra and E be an H -comodule algebra. There is an isomorphism of pointed sets*

$$\mathcal{H}^1(H, E) \cong \text{Tors}(H, E).$$

Proof: We construct a map $\tilde{\mathcal{T}} : \mathcal{Z}^1(H, E) \longrightarrow \text{tors}(H, E)$ in the following way. For any Hopf 1-cocycle X , let $\tilde{\mathcal{T}}(X)$ be the (H, E) -Hopf module (E, Δ_E^X) defined in § 1.5. It is clearly a torsor (indeed T^\bullet contains for example the unit of E). By Proposition 1.7, the map $\tilde{\mathcal{T}}$ induces a map $\mathcal{T} : \mathcal{H}^1(H, E) \longrightarrow \text{Tors}(H, E)$ on the quotients.

The injectivity of \mathcal{T} is a direct consequence of Proposition 1.7. Let us prove that \mathcal{T} is surjective. Take a torsor (T, Δ_T) and $u \in T^\bullet$. By definition, $\Delta_T(u)$ belongs to $(T \otimes H)^\times$. Applying the map $\vartheta_{u \otimes 1}^{-1} = \vartheta_u^{-1} \otimes \text{id}_H$, we define the element

$$X_T = (\vartheta_{u \otimes 1}^{-1} \circ \Delta_T)(u) = ((\vartheta_u^{-1} \otimes \text{id}_H) \circ \Delta_T)(u),$$

which belongs to $(E \otimes H)^\times$. Writing $\Delta_T(u) = u_0 \otimes u_1$, one gets $X_T = \vartheta_u^{-1}(u_0) \otimes u_1$. Let us compute the product $(u \otimes 1 \otimes 1)(d^2(X_T)d^0(X_T))$. First remark that we have the equalities

$$(u \otimes 1 \otimes 1)d^2(X_T) = (\vartheta_u \vartheta_u^{-1} \otimes \text{id}_H)(\Delta_T(u)) \otimes 1 = \Delta_T(u) \otimes 1.$$

On the other hand, we write

$$d^0(X_T) = (\Delta_E \circ \vartheta_u^{-1} \otimes \text{id}_H)(\Delta_T(u)) = \Delta_E(\vartheta_u^{-1}(u_0)) \otimes u_1.$$

By multiplying the two expressions, we get

$$\begin{aligned} (u \otimes 1 \otimes 1)d^2(X_T)d^0(X_T) &= \Delta_T(u)\Delta_E(\vartheta_u^{-1}(u_0)) \otimes u_1 \\ &= \Delta_T(u\vartheta_u^{-1}(u_0)) \otimes u_1 \\ &= \Delta_T(u_0) \otimes u_1 \\ &= u_0 \otimes \Delta_H(u_1). \end{aligned}$$

Finally we obtain

$$d^2(X_T)d^0(X_T) = \vartheta_u^{-1}(u_0) \otimes \Delta_H(u_1) = (\text{id}_E \otimes \Delta_H)(X_T) = d^1(X_T).$$

Hence X_T is a Hopf 1-cocycle.

We show now that the torsors (T, Δ_T) and $\tilde{T}(X_T) = (E, \Delta_E^{X_T})$ are equivalent. The wished isomorphism of Hopf modules between $(E, \Delta_E^{X_T})$ and (T, Δ_T) is given by the map ϑ_u . Indeed, for any element $x \in E$, one has the equalities

$$\begin{aligned} ((\vartheta_u \otimes \text{id}_H) \circ \Delta_E^{X_T})(x) &= (\vartheta_u \otimes \text{id}_H)(X_T \Delta_E(x)) \\ &= (\vartheta_u \otimes \text{id}_H)\left((\vartheta_u^{-1} \otimes \text{id}_H) \circ \Delta_T(u) \Delta_E(x)\right) \\ &= \Delta_T(u) \Delta_E(x) \\ &= \Delta_T(ux) \\ &= (\Delta_T \circ \vartheta_u)(x). \end{aligned}$$

□

Example 3.5. Let, as in §1.4, H_4 be the Sweedler four-dimensional Hopf algebra over a field k and E_2 be the algebra of dual numbers. The image of \tilde{T} in $\text{tors}(H, E)$ consists of the (H_4, E_2) -modules $T_{X_a} = (E_2, \Delta^{X_a})$ and $T_{Y_a} = (E_2, \Delta^{Y_a})$, where a runs through k and where the coactions are explicitly given by

$$\begin{aligned} \Delta^{X_a}(1) &= X_a = 1 \otimes 1 + a(1 \otimes h) - a(h \otimes 1) + a(h \otimes g) - a^2(h \otimes h) \\ \Delta^{X_a}(h) &= X_a \Delta(h) = 1 \otimes h + h \otimes g - a(h \otimes h) \end{aligned}$$

and

$$\begin{aligned} \Delta^{Y_a}(1) &= Y_a = 1 \otimes g + a(1 \otimes gh) - a(h \otimes g) + a(h \otimes 1) - a^2(h \otimes gh) \\ \Delta^{Y_a}(h) &= Y_a \Delta(h) = 1 \otimes gh + h \otimes 1 - a(h \otimes gh). \end{aligned}$$

Up to isomorphism, only two equivalence classes of torsors remain: those consisting in the class of (E_2, Δ) itself and the class of (E_2, Δ') , where

$$\Delta'(1) = 1 \otimes g \quad \text{and} \quad \Delta'(h) = h \otimes 1 + 1 \otimes gh.$$

Remark 3.6: Suppose that the algebras E and H are both commutative. Let T and T' be two (H, E) -Hopf torsors. Endow T' with the symmetric E -bimodule action. One may easily check that the tensor product $T \otimes_E T'$ is also an (H, E) -Hopf torsor with coaction given by $\Delta_{T \otimes_E T'}(t \otimes t') = t_0 \otimes t'_0 \otimes t_1 t'_1$. Indeed the set $(T \otimes_E T')^\bullet$ contains all the elements $u \otimes u'$, where u belongs to T^\bullet and u' to T'^\bullet . Whence $\text{tors}(H, E)$ is a monoid with product \otimes_E . Under these hypothesis of commutativity, we already noticed that $\mathcal{Z}^1(H, E)$ and $\mathcal{H}^1(H, E)$ are groups (Remark 1.3(b)). The map $\tilde{T} : \mathcal{Z}^1(H, E) \longrightarrow \text{tors}(H, E)$ is then a morphism of monoids. Following Theorem 3.4, the product of $\text{tors}(H, E)$ induces a group structure on the quotient $\text{Tors}(H, E)$.

3.3. Comparison with the group case. Let us show how to relate Definition 3.2 to the usual notion of torsors. Given a finite group G and a G -group A , a (G, A) -group torsor is a non-empty left G -set P on which A acts on the right in a compatible way with the G -action and such that P is an affine space over A (see [10]). Denote by $\text{Tors}(G, A)$ the set of isomorphism classes of (G, A) -group torsors, which is known to be isomorphic to $H^1(G, A)$ (Proposition I.33 in [10]). If P is a (G, A) -torsor, its class in $\text{Tors}(G, A)$ is written $[P]$.

Proposition 3.7: *Let G be a finite group, let k^G be the Hopf algebra of the functions on G , and E be an k^G -comodule algebra. For any (k^G, E) -Hopf torsor T , the set T^\times is a (G, E^\times) -group torsor.*

Proof: As previously observed, T^\times is an E^\times -set. By § 1.3, the group E^\times is equipped with a G -group structure. In the same way, if one writes $\Delta_T(u) = \sum_{g \in G} {}^g u \otimes \delta_g$ for $u \in T$, one deduces an

action of the group G on the set T . By Lemma 3.3, if u belongs to T^\times , then the element $\Delta_T(u)$ belongs to $(T \otimes k^G)^\times$, which is easily seen to be isomorphic to $(T^\times)^{|G|}$. So, for any $g \in G$, the element ${}^g u$ belongs to T^\times , hence T^\times is a G -group. The compatibility of the two G -structures on E^\times and T^\times is a consequence of the (k^G, E) -Hopf module structure of T . The fact that T^\times is an affine space over E^\times is precisely the bijectivity of ϑ_u proved in Lemma 3.1 for any $u \in T^\times$. \square

Denote by $c : \text{tors}(k^G, E) \longrightarrow \text{Tors}(G, E^\times)$ the map defined for any (k^G, E) -torsor T by

$$c(T) = [T^\times].$$

Corollary 3.8. *Let G be a finite group and E be a k^G -comodule algebra. The map c induces a bijection of pointed sets*

$$\text{Tors}(k^G, E) \cong \text{Tors}(G, E^\times).$$

Proof: The isomorphism is a direct consequence of Theorem 1.5, Theorem 3.4 of this article, and Proposition I.33 in [10]. It is given by the sequence of isomorphisms

$$\text{Tors}(k^G, E) \cong \mathcal{H}^1(k^G, E) \cong H^1(G, E^\times) \cong \text{Tors}(G, E^\times).$$

Let T be a (k^G, E) -torsor and u an element of T^\times . The sequence of isomorphisms associates to T the class of the (G, E^\times) -torsor E_T^\times defined as follows. As a set E_T^\times is nothing but E^\times . It is endowed with the G -action given for $g \in G$ and $x \in E^\times$ by

$$g \rightharpoonup x = \vartheta_u^{-1}({}^g u) {}^g x.$$

One verifies that $[E_T^\times] = [T^\times] = c(T)$ in $\text{Tors}(G, E^\times)$ via the isomorphisms $\vartheta_u : E^\times \longrightarrow T^\times$. \square

3.4. Comparison with the restricted case. Let H be a Hopf algebra, S an H -comodule algebra, and M an (H, S) -Hopf module. Recall that what we called M -torsor in [8] is a triple (X, Δ_X, β) , where $\Delta_X : X \longrightarrow X \otimes H$ is a map conferring X a structure of (H, S) -Hopf module and $\beta : M \longrightarrow X$ is an S -linear isomorphism. Here we rename this datum a *restricted M -torsor*. The set of restricted M -torsors is pointed with distinguished point $(M, \Delta_M, \text{id}_M)$. Two restricted M -torsors (X, Δ_X, β) and $(X', \Delta_{X'}, \beta')$ are *equivalent* if there exists $f \in \text{Aut}_S(M)$ such that the composition $\beta \circ f \circ \beta'^{-1} : X' \longrightarrow X$ is a morphism of (H, S) -Hopf modules. Denote by $\text{Tors}(M)$ the set of equivalence classes of restricted M -torsors; it is pointed with distinguished point the class of $(M, \Delta_M, \text{id}_M)$. By Theorem 3.4 and Theorem 2.6 of the present article, by Proposition 2.8 and Theorem 3.1 of [8], one deduces the following statement:

Corollary 3.9. *Let H be a Hopf algebra, S be an H -comodule algebra, and M be an (H, S) -Hopf module such that Condition (\mathcal{F}_n) is satisfied for $0 \leq n \leq 2$. Then there is a bijection of pointed sets*

$$\text{Tors}(M) \cong \text{Tors}(H, \text{End}_S(M)).$$

This result shows that, under weak technical conditions on M , the possible structures of $(H, \text{End}_S(M))$ -Hopf module on $\text{End}_S(M)$ are closely related to the possible (H, S) -Hopf-module

structures on M . More precisely, if $\text{End}_S(M)$ is equipped with an $(H, \text{End}_S(M))$ -Hopf module structure Δ , then following the track of Δ along the four isomorphisms

$$\text{Tors}(H, \text{End}_S(M)) \cong \mathcal{H}^1(H, \text{End}_S(M)) \cong H^1(H, M) \cong \text{Tors}(M),$$

one gets an (H, S) -Hopf-module structure Δ' on M defined on an element $m \in M$ by

$$\Delta'(m) = \varphi_0(m_0) \otimes \varphi_1 m_1.$$

Here we denote by $\varphi_0 \otimes \varphi_1$ the element $\Delta(\text{id}_M) \in \text{End}_S(M) \otimes H$, and as usual, we adopt the convention $\Delta_M(m) = m_0 \otimes m_1$.

REFERENCES

- [1] A. BLANCO FERRO, Hopf algebras and Galois descent, *Publ. Sec. Mat. Universitat Autònoma Barcelona* **30** (1986), n° 1, 65 – 80.
- [2] Ch. KASSEL, *Quantum Groups*, Graduate Texts in Mathematics 155, Springer-Verlag, New York (1995).
- [3] H. F. KREIMER, M. TAKEUCHI, Hopf algebras and Galois extensions of an algebra, *Indiana Univ. Math. J.* **30** (1981), n° 5, 675 – 692.
- [4] S. LANG, J. TATE, Principal homogeneous spaces over abelian varieties, *Amer. J. Maths.* **80** (1958), 659 – 684.
- [5] L. LE BRUYN, M. VAN DEN BERGH, F. VAN OYSTAEYEN, *Graded orders*, Birkhäuser, Boston – Basel (1988).
- [6] J.-L. LODAY, *Cyclic homology*, Grundlehren der Mathematischen Wissenschaften 301, Springer-Verlag, Berlin (1988).
- [7] Ph. NUSS, Noncommutative descent and non-abelian cohomology, *K-Theory* **12** (1997), n° 1, 23 – 74.
- [8] Ph. NUSS, M. WAMBST, Non-Abelian Hopf Cohomology, *J. Algebra* **312**, (2007), n° 2, 733 – 754.
- [9] J.-P. SERRE, *Corps locaux*, Troisième édition corrigée, Hermann, Paris (1968).
- [10] J.-P. SERRE, *Galois cohomology*, Springer-Verlag, Berlin – Heidelberg (1997). Translated from *Cohomologie galoisienne*, Lecture Notes in Mathematics 5, Springer-Verlag, Berlin – Heidelberg – New York (1973).
- [11] M. E. SWEEDLER, Cohomology of algebras over Hopf algebras, *Trans. Amer. Math. Soc.* **133** (1968), 205 – 239.